

ON AFFINITY RELATING TWO POSITIVE MEASURES AND THE CONNECTION COEFFICIENTS BETWEEN POLYNOMIALS ORTHOGONALIZED BY THESE MEASURES

PAWEŁ J. SZABŁOWSKI

ABSTRACT. We consider two positive, normalized measures $dA(x)$ and $dB(x)$ related by the relationship $dA(x) = \frac{C}{x+D}dB(x)$ or by $dA(x) = \frac{C}{x^2+E}dB(x)$ and $dB(x)$ is symmetric. We show that then the polynomials sequences $\{a_n(x)\}$, $\{b_n(x)\}$ orthogonal with respect to these measures are related by the relationship $a_n(x) = b_n(x) + \kappa_n b_{n-1}(x)$ or by $a_n(x) = b_n(x) + \lambda_n b_{n-2}(x)$ for some sequences $\{\kappa_n\}$ and $\{\lambda_n\}$. We present several examples illustrating this fact and also present some attempts for generalizations. We also give some universal identities involving polynomials $\{b_n(x)\}$ and the sequence $\{\kappa_n\}$.

1. INTRODUCTION

We study relationship between the pair of orthogonal polynomials and the pair of measures that make these polynomials orthogonal. This problem has practical importance. If solved in full generality would enable quick and easy way of finding sets of orthogonal polynomials for a given measure simplifying the usual path of the Gram-Smith orthogonalization. Besides it would provide quick and easy way of finding 'connection coefficients' between the two analyzed orthogonal polynomials. On its side 'connection coefficients', as it is well known supply many useful informations about the properties of the involved sets of polynomials. So far in the literature devoted to connection coefficients like [5], [6], [7] the authors studied the properties of these coefficients and their relationship to zeros of orthogonal polynomials in question without referring to the properties of orthogonalizing measures.

We are solving the problem of affinity between connection coefficients and measures that make polynomials orthogonal only partially. There are still many challenging questions that we pose in Section 4 and which are unsolved to our knowledge.

To be more precise we will assume throughout the paper the following setting:

We consider two sequences of monic, orthogonal polynomials $\{a_n\}$ and $\{b_n\}$ such that their 3-term recurrences are as given below:

$$(1.1) \quad a_{n+1}(x) = (x - \alpha_n)a_n(x) - \hat{\alpha}_{n-1}a_{n-1}(x),$$

$$(1.2) \quad b_{n+1}(x) = (x - \beta_n)b_n(x) - \hat{\beta}_{n-1}b_{n-1}(x)$$

with $a_{-1}(x) = b_{-1}(x) = 0$, $a_0(x) = b_0(x) = 1$.

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In [4], Proposition 1 it was shown that if these measures are such that if $\text{supp } A = \text{supp } B$ and $dA(x) = \frac{1}{P_r(x)}dB(x)$, where P_r is a polynomial of order r , then there exist r sequences $\left\{c_n^{(j)}\right\}_{n \geq 1, 1 \leq j \leq r}$ such that

$$a_n(x) = b_n(x) + \sum_{j=1}^r c_n^{(j)} b_{n-j}(x).$$

In the cited result it was not presented how to relate 3-term recurrence satisfied by say the set $\{b_n\}$ and the form of the polynomial P_r to the form of the coefficients $\left\{c_n^{(j)}\right\}_{n \geq 1, 1 \leq j \leq r}$.

In the present paper we continue the research started in [4] and relate the 3-term recurrence coefficients satisfied by $\{b_n\}$ and the exact form of the polynomial P_r for $r = 1, 2$ to coefficients $c_n^{(1)}$ for $r = 1$ and $c_n^{(1)}, c_n^{(2)}$ for $r = 2$. We also give the 3-term recurrence coefficients of the polynomials $\{a_n\}$. Besides we also provide certain universal identities satisfied by the polynomials $\{b_n\}$, coefficients $\{\hat{\beta}_n\}$, $\{c_n^{(1)}\}$ and the parameters of the polynomials P_1 .

The paper is organized as follows: In the next Section 2 we present our main result concerning the case $r = 1$ and illustrate it by 3 examples concerning well known families of polynomials like Jacobi or Charlier. It is presented in Section 3. Less complete or less simple and nice results are presented in Section 4. Here also we will also illustrate the developed ideas by a few examples. Finally Section 5 contains less interesting or lengthy proofs of our results.

2. MAIN RESULTS

The simplest but also the most important case is when $r = 1$. This case is treated by the theorem below:

Theorem 1. *Let the sequence of monic, orthogonal polynomials $\{b_n\}$ be defined by the 3-term recurrence (1.2). Suppose that $dB(x)$ is the positive measure that makes these polynomials orthogonal. Let us consider another normalized measure $dA(x)$ related to $dB(x)$ by the relationship:*

$$(2.1) \quad dA(x) = \frac{C}{x+D} dB(x),$$

so that $\frac{C}{(x+D)} \geq 0$ on the support of B .

Then there exists a number sequence $\{\kappa_n\}$ defined by the relationship:

$$\kappa_n = \beta_{n-1} - \frac{\hat{\beta}_{n-2}}{\kappa_{n-1}} + D,$$

$n \geq 2$ with $\kappa_1 = \beta_0 + D - C$, such that the monic polynomial sequence defined by:

$$(2.2) \quad a_n(x) = b_n(x) + \kappa_n b_{n-1}(x).$$

satisfies the 3-term recurrence (1.1) with:

$$(2.3) \quad \alpha_n = \beta_n + \kappa_n - \kappa_{n+1},$$

$$(2.4) \quad \hat{\alpha}_{n-1} = \kappa_n \frac{\hat{\beta}_{n-2}}{\kappa_{n-1}},$$

and is orthogonal with respect to the measure $dA(x)$.

Proof. Is shifted to section 5. \square

We have immediate remarks, observations and corollaries

Remark 1. All coefficients κ_n have the same sign i.e. are either all positive or all negative. This follows the fact that since $dA(x)$ and $dB(x)$ are positive measures we must have nonnegative both $\hat{\alpha}_n$ and $\hat{\beta}_n$. Then we use (2.4).

Remark 2. Notice that following relationship $\kappa_1 = \beta_0 + D - C$ and (2.2), $\frac{C}{x+D}$ can be written as $\frac{1}{a_1(x)/C+1}$ which fits assumptions of Proposition 1 of [4].

Remark 3. Following (2.4) and the fact that $\hat{\alpha}_{n-1} \geq 0$ we deduce that either $\beta_n + D \geq 0$ or $\beta_n + D \leq 0$ for all $n \geq 1$. Consequently either we have for all $n \geq 0$

$$\beta_{n-1} + D \leq \kappa_n \leq 0 \text{ or } 0 \leq \kappa_n \leq \beta_{n-1} + D.$$

Corollary 1. Under assumptions of Theorem 1 we have

$$(2.5) \quad b_n(x) = a_n(x) + \sum_{j=1}^n (-1)^j \left(\prod_{k=n-j+1}^n \kappa_k \right) a_{n-j}(x)$$

for $n = 0, 1, 2, \dots$. Further under additional assumption that $\int_{\text{supp } B} \frac{1}{(x+D)^2} dB(x) < \infty$ we have:

$$1 + \sum_{n \geq 1} \left(\prod_{k=1}^n \frac{\kappa_k}{\hat{\beta}_{k-1}} \right)^2 = C^2 \int_{\text{supp } B} \frac{1}{(x+D)^2} dB(x)$$

and

$$(2.6) \quad \frac{C}{x+D} = 1 + \sum_{n \geq 1} (-1)^n \left(\prod_{k=1}^n \frac{\kappa_k}{\hat{\beta}_{k-1}} \right) b_n(x),$$

on $\text{supp } B$ in $L_2(\text{supp } B, \mathcal{B}, dB(x))$. If additionally $\sum_{n \geq 1} \left(\prod_{k=1}^n \frac{\kappa_k}{\hat{\beta}_{k-1}} \right)^2 \log^2 n < \infty$, then convergence in (2.6) is almost $(dB(x))$ pointwise on $\text{supp } B$.

Proof. Using (2.5) we have $b_n(x) + \kappa_n b_{n-1}(x) = a_n + \sum_{j=1}^n (-1)^j \left(\prod_{k=n-j+1}^n \kappa_k \right) a_{n-j}(x) + \kappa_n a_{n-1} + \sum_{j=1}^{n-1} (-1)^j \left(\prod_{k=n-j}^n \kappa_k \right) a_{n-1-j}(x) = a_n$. Now we apply idea of ratio of density expansion presented in [4] and use (2.5). On the way we notice that the requirement that both measures (i.e. dA and dB) have densities with respect to Lebesgue measure can be dropped. We also utilize the fact that $\int_{\text{supp } B} b_n^2(x) dB(x) = \prod_{k=0}^{n-1} \hat{\beta}_k$ which follows Favard's Theorem. Then we also use Rademacher–Menshov theorem concerning almost sure convergence of L_2 converging Fourier series. \square

3. EXAMPLES

To illustrate how simple and easy is to utilize the presented in the previous section observations and rules let us consider the following few examples:

Example 1 (Jacobi polynomials). Recall (e.g. basing on [2] or [1]) that monic Jacobi polynomials $J_n^{(\alpha, \beta)}(x)$ satisfy the following 3-term recurrence:

$$(3.1a) \quad J_{n+1}^{(\alpha, \gamma)}(z) = \left(x + \frac{\alpha^2 - \gamma^2}{(2n + \alpha + \gamma + 2)(\alpha + \gamma + 2n)} \right) J_n^{(\alpha, \gamma)}(x)$$

$$(3.1b) \quad - \frac{4n(\alpha + \gamma + n)(n + \alpha)(n + \gamma)}{(\alpha + \gamma + 2n - 1)(2n + \alpha + \gamma)^2(\alpha + \gamma + 2n + 1)} J_{n-1}^{(\alpha, \gamma)}(z).$$

Besides one knows also that the normalized measure that makes these polynomials orthogonal is the following:

$$f(x; \alpha, \gamma) = \frac{\Gamma(\alpha + \gamma + 2)}{2^{\alpha + \gamma + 1} \Gamma(\gamma + 1) \Gamma(\alpha + 1)} (1 - x)^\alpha (1 + x)^\gamma,$$

where $\Gamma(\eta)$ denotes value of the Gamma function at η , for $|x| < 1$ and $\alpha, \beta > -1$.

Now let us take $\alpha > 0, \beta > -1$, $dB(x) = f(x; \alpha, \beta) dx$ and $dA(x) = f(\alpha - 1, \beta) dx$, $b_n(x) = J_n^{(\alpha, \gamma)}(x)$ and $a_n(x) = J_n^{(\alpha - 1, \gamma)}(x)$. One can easily notice that $dA(x) = \frac{C}{(-1+x)} dB(x)$, where $C = -\frac{2\alpha}{\alpha + \gamma + 1}$, hence $D = -1$. From (3.1) it follows also that

$$(3.2) \quad \beta_n = -\frac{\alpha^2 - \gamma^2}{(2n + \alpha + \gamma + 2)(\alpha + \gamma + 2n)},$$

$$(3.3) \quad \hat{\beta}_{n-1} = \frac{4n(\alpha + \gamma + n)(n + \alpha)(n + \gamma)}{(\alpha + \gamma + 2n - 1)(2n + \alpha + \gamma)^2(\alpha + \gamma + 2n + 1)}.$$

Thus $\kappa_1 = \beta_0 + D - C = -\frac{\alpha^2 - \gamma^2}{(\alpha + \gamma + 2)(\alpha + \gamma)} + \frac{2\alpha}{\alpha + \gamma + 1} - 1 = -2\frac{\gamma + 1}{(\alpha + \gamma + 1)(\alpha + \gamma + 2)}$ and consequently coefficients κ_n satisfy recursive equation:

$$\kappa_n = \beta_{n-1} - 1 - \frac{\hat{\beta}_{n-2}}{\kappa_{n-1}},$$

for $n \geq 2$. One can also easily notice that

$$(3.4) \quad \kappa_n = -\frac{2n(n + \gamma)}{(\alpha + \gamma + 2n)(\alpha + \gamma + 2n - 1)}$$

satisfies above mentioned recursive equation. Hence

$$J_n^{(\alpha - 1, \gamma)}(x) = J_n^{(\alpha, \gamma)}(x) + \kappa_n J_{n-1}^{(\alpha, \gamma)}(x).$$

As far as application of Corollary 1 is concerned we have the following identity true for $\alpha > 1, \gamma > -1$, and almost all $|x| < 1$:

$$1 = \frac{\alpha + \gamma + 1}{2\alpha} (1 - x) \left(1 + \sum_{n \geq 1} \frac{(\alpha + \gamma + 1)_{2n}}{2^n (\alpha + \gamma + 1)_n (\alpha + 1)_n (\alpha + \gamma + 1)_n} J_n^{(\alpha, \gamma)}(x) \right),$$

where we use the so called Pochhammer symbol $(a)_n = a(a + 1) \dots (a + n - 1)$.

This is so since $\frac{\kappa_n}{\gamma_{n-1}} = -\frac{(\alpha + \gamma + 2n + 1)(\alpha + \gamma + 2n)}{2(\alpha + n)(\alpha + \gamma + n)}$, by (3.3) and (3.4) and because $\int_{-1}^1 \frac{1}{(1-x)^2} dB(x) < \infty$ for $\gamma > -1, \alpha - 2 > -1$.

Example 2 (Charlier polynomials). Basing on [3] let us recall that monic Charlier polynomials $\{c_n(x; \lambda)\}_{n \geq -1}$ are polynomials given by the following 3-term recurrence

$$c_{n+1}(x; \lambda) = (x - n - \lambda)c_n(x; \lambda) - n\lambda c_{n-1}(x; \lambda),$$

with $c_{-1}(x; \lambda) = 0$, $c_0(x; \lambda) = 1$. They are orthogonal with respect to discrete measure concentrated at nonnegative integers with mass at n equal to $\exp(-\lambda) \frac{\lambda^n}{n!}$, $n \geq 0$. Another words this measure is the Poisson normalized measure.

In order not to complicate too much let us take $dB(n) = \exp(-\lambda) \frac{\lambda^n}{n!}$ and $dA(n) = \frac{C}{n+1} dB(n)$ for $n = 0, 1, \dots$. Since $\sum_{n \geq 0} \frac{\lambda^n}{(n+1)!} = \frac{(\exp(\lambda)-1)}{\lambda}$ we see that $C = \frac{\lambda \exp(\lambda)}{\exp(\lambda)-1}$. Naturally we have also $D = 1$ and $\beta_n = n + \lambda$ and $\hat{\beta}_{n-1} = n\lambda$, hence $\kappa_1 = \lambda + 1 - \frac{\lambda \exp(\lambda)}{-1 + \exp(\lambda)} = \frac{\exp(\lambda)-1-\lambda}{e^\lambda-1}$. Thus recursive equation satisfied by coefficients κ_n is the following:

$$\kappa_n = n + \lambda - \frac{(n-1)\lambda}{\kappa_{n-1}},$$

$n \geq 2$. In particular we have $\kappa_2 = 2 \frac{\exp(\lambda)-1-\lambda-\lambda^2/2}{\exp(\lambda)-1-\lambda}$, $\kappa_3 = 3 \frac{\exp(\lambda)-1-\lambda-\lambda^2/2-\lambda^3/3!}{\exp(\lambda)-1-\lambda-\lambda^2/2}$ and in general it is easy to see that

$$\kappa_n = n \frac{\exp(\lambda) - \sum_{j=0}^n \frac{\lambda^j}{j!}}{\exp(\lambda) - \sum_{j=0}^{n-1} \frac{\lambda^j}{j!}} = n \frac{\sum_{j \geq n+1} \frac{\lambda^j}{j!}}{\sum_{j \geq n} \frac{\lambda^j}{j!}}.$$

Thus we have in particular

$$\begin{aligned} a_n(x) &= c_n(x) + \kappa_n c_{n-1}(x), \\ \alpha_n &= n + \lambda + \kappa_n - \kappa_{n+1}, \\ \hat{\alpha}_{n-1} &= \frac{\lambda n (\sum_{j \geq n+1} \frac{\lambda^j}{j!}) (\sum_{j \geq n-1} \frac{\lambda^j}{j!})}{(\sum_{j \geq n} \frac{\lambda^j}{j!})^2}. \end{aligned}$$

As far as application of Corollary 1 is concerned we have the following identity true for $\lambda > 0$ and $x = 0, 1, \dots$

$$(3.5) \quad 1 = \frac{e^\lambda - 1}{\lambda e^\lambda} (1+x) \left(1 + \sum_{n \geq 1} (-1)^n \frac{e^\lambda - \sum_{k=0}^n \lambda^k / k!}{\lambda^n (e^\lambda - 1)} c_n(x; \lambda) \right).$$

This is so since $\frac{\kappa_n}{\beta_{n-1}} = \frac{\exp(\lambda) - \sum_{j=0}^n \frac{\lambda^j}{j!}}{\lambda (\exp(\lambda) - \sum_{j=0}^{n-1} \frac{\lambda^j}{j!})}$. Let us observe that (3.5) is not satisfied for non-positive integer x .

Example 3 (Legendre polynomials). As it is known Legendre polynomials are the Jacobi polynomials with $\alpha, \gamma = 0$. As $dB(x)$ let us consider measure with the density $f(x; 0, 0) = 1/2$ on $[-1, 1]$. As $dA(x)$ let us consider measure with the density $\frac{C}{2(3-x)}$ on $[-1, 1]$. Parameter C we get by direct integration, namely $C = \frac{-2}{\ln 2}$ while $D = -3$. Further using (3.2) and (3.3) we get:

$$\beta_n = 0, \hat{\beta}_{n-1} = \frac{n^2}{(2n-1)(2n+1)}.$$

Hence $\kappa_1 = 0 - 3 + \frac{2}{\ln 2}$ and consequently coefficients κ_n are given by the following recursive equation:

$$\kappa_{n+1} = -3 - \frac{n^2}{(2n-1)(2n+1)\kappa_n},$$

for $n \geq 1$. In particular we get $\kappa_2 = -3 - \frac{1}{3(-3+2/\ln 2)} = -\frac{(26 \ln 2 - 18)}{9 \ln 2 - 6}$. Finally we deduce that polynomials defined by

$$a_n(x) = J_n^{(0,0)} + \kappa_n J_{n-1}^{(0,0)}(x),$$

are orthogonal with respect to the measure with the density: $\frac{2}{(3-x)\ln 2}$ on $[-1, 1]$.

4. EXTENSIONS AND OPEN PROBLEMS

In this section we are going to present some generalizations of the results of the Section 2. The results are not as nice and compact as the ones presented above that is why we present them here. We will also pose some open problems that appeared immediately when writing the article.

Let us return to the setting that was presented in the Introduction and consider the case $r = 2$. Let us assume that measures dA and dB are related to one another by the relationship

$$(4.1) \quad dA(x) = \frac{C}{x^2 + Dx + E} dB(x)$$

and that constants C, D, E are such that $\frac{C}{x^2 + Dx + E} \geq 0$ on $\text{supp } B$ and that measure dA is normalized. Following cited already [4], Proposition 1 we deduce that then polynomials $\{a_n\}$ and $\{b_n\}$ orthogonal with respect to these measures are related by the relationship

$$(4.2) \quad a_n(x) = b_n(x) + \kappa_n b_{n-1}(x) + \lambda_n b_{n-2}(x),$$

for some number sequences $\{\kappa_n\}$ and $\{\lambda_n\}$ given in the Proposition below:

Proposition 1. *Suppose normalized, positive measures dA and dB are related to one another by (4.1). Let further respectively polynomial sequences $\{a_n\}$ and $\{b_n\}$ orthogonal with respect to these measures satisfy 3-term recurrence (1.1) and (1.2). Then there exist two number sequences $\{\kappa_n\}$ and $\{\lambda_n\}$ such that (4.2) is satisfied. Moreover number sequences $\{\kappa_n\}$, $\{\lambda_n\}$, $\{\alpha_n\}$, $\{\hat{\alpha}_n\}$, $\{\beta_n\}$, $\{\hat{\beta}_n\}$ are related to one another by the system of equations:*

$$(4.3) \quad \kappa_{n+1} + \alpha_n = \beta_n + \kappa_n,$$

$$(4.4) \quad \lambda_{n+1} + \alpha_n \kappa_n + \hat{\alpha}_{n-1} = \hat{\beta}_{n-1} + \kappa_n \beta_{n-1} + \lambda_n,$$

$$(4.5) \quad \alpha_n \lambda_n + \hat{\alpha}_{n-1} \kappa_{n-1} = \kappa_n \hat{\beta}_{n-2} + \lambda_n \beta_{n-2},$$

$$(4.6) \quad \hat{\alpha}_{n-1} \lambda_{n-1} = \lambda_n \hat{\beta}_{n-3}.$$

with $\lambda_1 = 0$ and κ_1, κ_2 and λ_2 defined as solutions of the system of 7 equations $0 = \int_{\text{supp } B} (b_1(x) + \kappa_1) dA(x) = \int_{\text{supp } B} (b_2(x) + \kappa_2 b_1(x) + \lambda_2) dA(x) = \int_{\text{supp } B} (b_2(x) + \kappa_2 b_1(x) + \lambda_2) (b_1(x) + \kappa_1) dA(x) = \int_{\text{supp } B} b_1(x) (x^2 + Dx + E) dA(x) = \int_{\text{supp } B} b_2(x) (x^2 + Dx + E) dA(x), \int_{\text{supp } B} (x^2 + Dx + E) dA(x) = C, \int_{\text{supp } B} b_1^2(x) (x^2 + Dx + E) dA(x) = \hat{\beta}_0$ with 4 additional unknowns $\int_{\text{supp } B} b_1(x) dA(x), \int_{\text{supp } B} b_2(x) dA(x), \int_{\text{supp } B} b_1(x) b_2(x) dA(x), \int_{\text{supp } B} b_2^2(x) dA(x)$.

Proof. Uninteresting proof is shifted to Section 5. \square

Now let us simplify calculations by assuming that the measure dB and polynomial $(x^2 + Dx + E)$ are symmetric which implies that polynomials orthogonal with respect to dB (i.e. b_n) must contain only either even or odd powers of x . Hence coefficients β_n are equal to zero for $n \geq 0$ and also that $D = 0$. Consequently measure dA must also be symmetric and by similar argument we deduce that coefficients $\alpha_n = 0$ for $n \geq 0$. This also results in the fact that coefficients κ_n are also zero for all $n \geq 0$.

As a result we have the following Lemma which is in fact a corollary of the Proposition 1.

Lemma 1. *Suppose normalized, positive measures dA and dB are related to one another by the relationship:*

$$dA(x) = \frac{C}{x^2 + E} dB(x).$$

Let further respectively polynomial sequences $\{a_n\}$ and $\{b_n\}$ orthogonal with respect to these measures satisfy 3-term recurrence (1.1) and (1.2). With $\beta_n = 0$ for $n \geq 0$. Then there exist a number sequences $\{\lambda_n\}$ such that

$$(4.7) \quad a_n(x) = b_n(x) + \lambda_n b_{n-2}(x),$$

and $\alpha_n = 0$ for $n \geq 0$. Moreover number sequence $\{\lambda_n\}$, satisfies the following second order recursive equation for $n \geq 3$:

$$(4.8) \quad \lambda_{n+1} = \lambda_n + \hat{\beta}_{n-1} - \frac{\lambda_n}{\lambda_{n-1}} \hat{\beta}_{n-3},$$

with $\lambda_1 = 0$, $\lambda_2 = \hat{\beta}_0 + E - C$, $\lambda_3 = \hat{\beta}_1 + E - \frac{E}{C-E} \hat{\beta}_0$.

Coefficients $\hat{\alpha}_n$ are given by relationship:

$$\hat{\alpha}_n = \frac{\lambda_{n+1}}{\lambda_n} \hat{\beta}_{n-2}.$$

Proof. We apply assumptions to the system of equations (4.3-4.6) getting:

$$\begin{aligned} \lambda_{n+1} + \hat{\alpha}_{n-1} &= \lambda_n + \hat{\beta}_{n-1}, \\ \hat{\alpha}_{n-1} \lambda_{n-1} &= \lambda_n \hat{\beta}_{n-3}, \end{aligned}$$

from which we get (4.8). To get initial conditions we notice that $a_2(x) = x^2 - \hat{\beta}_0 + \lambda_2 = x^2 + E + (\lambda_2 - \beta_0 - E)$, so from the relationships $\int_{\text{supp } B} a_2(x) dA(x) = 0$ and $\int_{\text{supp } B} (x^2 + E) dA(x) = C$ we get $C + (\lambda_2 - \beta_0 - E) = 0$. Now to get λ_3 we use relationship: $\int_{\text{supp } B} a_1(x) a_3(x) dA(x) = 0$, using on the way the fact that $a_3(x) = b_3(x) + \lambda_3 x = x(x^2 - \hat{\beta}_0) - \hat{\beta}_1 x + \lambda_3 x = x^3 + x(\lambda_3 - \hat{\beta}_0 - \hat{\beta}_1)$ and that $\int_{\text{supp } B} (x^2 - \hat{\beta}_0)(x^2 + E) dA(x) = 0$. We have: $0 = \int_{\text{supp } B} ((x^2 - \hat{\beta}_0)(x^2 + E) + (\lambda_3 - \hat{\beta}_1 - E)x^2 + \hat{\beta}_0 E) dA(x) = \int_{\text{supp } B} ((\lambda_3 - \hat{\beta}_1 - E)(x^2 + E) + \hat{\beta}_0 E - E(\lambda_3 - \hat{\beta}_1 - E)) dA(x) = C(\lambda_3 - \hat{\beta}_1 - E) - E(\lambda_3 - \hat{\beta}_1 - E - \hat{\beta}_0) = 0$. \square

We will briefly illustrate this Lemma by the following example.

Example 4 (Jacobi polynomials revisited). *Let us consider symmetric case i.e. assuming that parameters α and γ are equal say to a . Then $\beta_n = 0$ and $\hat{\beta}_{n-1} = \frac{n(n+2a)}{(2a+2n-1)(2a+2n+1)}$. Parameters C and E are now equal to $-\frac{2a}{2a+1}$ and -1 respectively. Hence*

$$\begin{aligned} \lambda_2 &= \hat{\beta}_0 + E - C = -\frac{2}{(2a+1)(2a+3)} \\ \lambda_3 &= \hat{\beta}_1 + E - \frac{E}{C-E} \hat{\beta}_0 = -\frac{6}{(2a+3)(2a+5)}. \end{aligned}$$

Besides one can easily check that sequence $\left\{-\frac{n(n-1)}{(2a+2n-1)(2a+2n-3)}\right\}$ satisfies (4.8).

So $\lambda_n = -\frac{n(n-1)}{(2a+2n-1)(2a+2n-3)}$, $n \geq 2$. Hence we have:

$$J_n^{(\alpha-1, \alpha-1)}(x) = J_n^{(\alpha, \alpha)} - \frac{n(n-1)}{(2a+2n-1)(2a+2n-3)} J_{n-2}^{(\alpha, \alpha)}.$$

In particular notice that for $\alpha = 1/2$ we have $J_n^{(1/2, 1/2)}(x) = U_n(x)/2^n$ where U_n are the Chebyshev polynomials of the second kind, while $J_n^{(-1/2, -1/2)}(x) = T_n(x)/2^{n-1}$ for $n \geq 2$ and $J_n^{(-1/2, -1/2)}(x) = T_n(x)$ for $n = 0, 1$, where $T_n(x)$, are the Chebyshev polynomials of the first kind. Besides $\left[-\frac{n(n-1)}{(2a+2n-1)(2a+2n-3)}\right]_{\alpha=1/2} = -\frac{1}{4}$ and we end up with well known relationship between Chebyshev polynomials of the first and second kind:

$$T_n(x) = (U_n(x) - U_{n-2}(x))/2.$$

Remark 4. Notice that we could have reached the result in the above mentioned example by applying the procedure described in Section 2 twice once for monomial $1-x$ and then for $1+x$.

Remark 5. Notice also that one could invert relationship (4.7) and find connection coefficients of polynomials $\{b_n\}$ expressed in terms of polynomials $\{a_n\}$. Like in the setting of Corollary 1 they would be expressed in terms of products of coefficients $\{\lambda_n\}$ (in fact either only with odd or even numbers) and consequently obtain expansion similar (2.6) (in fact involving polynomials $\{b_n\}$ with even numbers).

4.1. Open problems.

- Is it possible to simplify equation (4.8) and reduce it to the first order recursive equation?
- Is it possible to simplify system of equations (4.3-4.6) and reduce it to the problem of solving system of the first order recursive equations?
- More generally is it possible to solve general system of equations presented in Proposition 1 of [4] or at least deduce more properties of coefficients $\left\{c_n^{(j)}\right\}_{n \geq 1, 1 \leq j \leq r}$ not only that for $j > r$ they are zeros.
- Is it possible give some properties of connection coefficients between two sets of orthogonal polynomials given the fact that orthogonalizing measures are related by the known relationship $dA(x) = F(x)dB(x)$ for functions F different from the reciprocal of a polynomial. It seems possible to consider rational functions on the first fire.

5. PROOFS

Proof of Theorem 1. Noticing that the proof of Proposition 1, iii) does not require the measures $dA(x)$ and $dB(x)$ to have densities we can apply its assertion and deduce that if positive, normalized measures are related by the relationship (2.1) then the polynomial sequences $\{a_n\}$ and $\{b_n\}$ orthogonal respectively with respect to these measures are related by (2.2). Hence sequence $\{\kappa_n\}$ exists and consequently we have $a_n(x) = b_n(x) + \kappa_n b_{n-1}(x)$. Remembering that sequences of polynomials $\{a_n\}$ and $\{b_n\}$ are orthogonal and satisfy the following 3-term recurrences:

$$\begin{aligned} a_{n+1}(x) &= (x - \alpha_n)a_n(x) - \hat{\alpha}_{n-1}a_{n-1}(x), \\ b_{n+1}(x) &= (x - \beta_n)b_n(x) - \hat{\beta}_{n-1}b_{n-1}(x). \end{aligned}$$

So on one hand we have

$$\begin{aligned} xa_n(x) &= a_{n+1}(x) + \alpha_n a_n(x) + \hat{\alpha}_{n-1} a_{n-1} \\ &= b_{n+1}(x) + (\kappa_{n+1} + \alpha_n \kappa_n) b_n + (\alpha_n \kappa_n + \hat{\alpha}_{n-1}) b_{n-1}(x) + \hat{\alpha}_{n-1} \kappa_{n-1} b_{n-2}(x). \end{aligned}$$

On the other we have:

$$\begin{aligned} xa_n(x) &= xb_n(x) + x\kappa_n b_{n-1}(x) \\ &= b_{n+1}(x) + (\beta_n + \kappa_n) b_n(x) + (\hat{\beta}_{n-1} + \kappa_n \beta_{n-1}) b_{n-1} + \kappa_n \hat{\beta}_{n-2} b_{n-2}(x). \end{aligned}$$

Hence we must have:

$$\begin{aligned} \kappa_{n+1} + \alpha_n &= \beta_n + \kappa_n, \\ \alpha_n \kappa_n + \hat{\alpha}_{n-1} &= \hat{\beta}_{n-1} + \kappa_n \beta_{n-1}, \\ \hat{\alpha}_{n-1} \kappa_{n-1} &= \kappa_n \hat{\beta}_{n-2} \end{aligned}$$

Now let us get α_n from the first of the equations

$$\alpha_n = \beta_n + \kappa_n - \kappa_{n+1}.$$

We get further

$$\hat{\alpha}_{n-1} = -\beta_n \kappa_n - \kappa_n^2 + \kappa_n \kappa_{n+1} + \hat{\beta}_{n-1} + \kappa_n \beta_{n-1}.$$

So finally we have:

$$\kappa_{n-1}(-\beta_n \kappa_n - \kappa_n^2 + \kappa_n \kappa_{n+1} + \hat{\beta}_{n-1} + \kappa_n \beta_{n-1}) = \kappa_n \hat{\beta}_{n-2}$$

dividing both sides by $\kappa_n \kappa_{n-1}$ we get:

$$\kappa_{n+1} = \kappa_n + \frac{\hat{\beta}_{n-2}}{\kappa_{n-1}} - \frac{\hat{\beta}_{n-1}}{\kappa_n} + \beta_n - \beta_{n-1}$$

Now notice that we can rearrange terms on both sides of this equation in the following way:

$$\kappa_{n+1} + \frac{\hat{\beta}_{n-1}}{\kappa_n} - \beta_n = \kappa_n + \frac{\hat{\beta}_{n-2}}{\kappa_{n-1}} - \beta_{n-1},$$

proving that quantity $\kappa_n + \frac{\hat{\beta}_{n-2}}{\kappa_{n-1}} - \beta_{n-1}$ does not depend on n and is equal to $\kappa_2 + \frac{\hat{\beta}_0}{\kappa_1} - \beta_1$.

We can easily find this quantity by finding directly quantities κ_1 and κ_2 .

Naturally we have $\kappa_0 = 1$. Remembering that $dB(x) = (d + cx)dA(x)$, that

$$b_{n+1}(x) = (x - \beta_n) b_n(x) - \hat{\beta}_n b_{n-1}(x)$$

and since $\int_{\text{supp } A} a_1(x) dA(x) = 0$ we must have

$$1 = \int_{\text{supp } A} dA(x) = \int_{\text{supp } A} \frac{C}{(D+x)} dB(x).$$

Now since $a_1(x) = b_1(x) + \kappa_1$ we have

$$\begin{aligned} 0 &= \int_{\text{supp } A} \frac{C(b_1(x) + \kappa_1)}{(D+x)} dB(x) \\ &= C + (\kappa_1 - \beta_0 - D) \int \frac{C}{x+D} dB(x) = C + \kappa_1 - \beta_0 - D, \end{aligned}$$

So

$$\kappa_1 = \beta_0 + D - C.$$

To find κ_2 we use the fact that $a_2(x) = b_2(x) + \kappa_2 b_1(x)$. Hence we have:

$$\begin{aligned} 0 &= C \int_{\text{supp } A} \frac{(b_2(x) + \kappa_2 b_1(x))}{(D+x)} dB(x) \\ &= C \int_{\text{supp } A} \frac{\left((-D - \beta_1 + \kappa_2)(b_1(x) + \kappa_1 - \kappa_1) - \hat{\beta}_0\right)}{(D+x)} dB(x) \\ &= C \int_{\text{supp } A} \frac{\left((D + \beta_1 - \kappa_2)\kappa_1 - \hat{\beta}_0\right)}{(D+x)} dB(x) = \left((D + \beta_1 - \kappa_2)\kappa_1 - \hat{\beta}_0\right). \end{aligned}$$

Hence we see that $\kappa_2 + \frac{\hat{\beta}_0}{\kappa_1} - \beta_1 = D$. \square

Proof of Proposition 1. Assuming that both sequences of polynomials i.e. $\{a_n\}$ and $\{b_n\}$ are orthogonal we have on one hand:

$$\begin{aligned} xa_n(x) &= a_{n+1} + \alpha_n a_n(x) + \hat{\alpha}_{n-1} a_{n-1}(x) = \\ &= b_{n+1}(x) + (\kappa_{n+1} + \alpha_n) b_n(x) + (\lambda_{n+1} + \alpha_n \kappa_n + \hat{\alpha}_{n-1}) b_{n-1}(x) \\ &\quad + (\alpha_n \lambda_n + \hat{\alpha}_{n-1} \kappa_{n-1}) b_{n-2}(x) + \hat{\alpha}_{n-1} \lambda_{n-1} b_{n-3}(x). \end{aligned}$$

and on the other:

$$\begin{aligned} xa_n(x) &= x(b_n(x) + \kappa_n b_{n-1}(x) + \lambda_n b_{n-2}(x)) \\ &= b_{n+1} + (\beta_n + \kappa_n) b_n(x) + \left(\hat{\beta}_{n-1} + \kappa_n \beta_{n-1} + \lambda_n\right) b_{n-1}(x) + \\ &\quad \left(\kappa_n \hat{\beta}_{n-2} + \lambda_n \beta_{n-2}\right) b_{n-2}(x) + \lambda_n \hat{\beta}_{n-3} b_{n-3}(x). \end{aligned}$$

Comparing expressions by b_n, b_{n-1}, b_{n-2} and b_{n-3} we get equations (4.3-4.6). \square

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DEPARTMENT OF MATHEMATICS AND INFORMATION SCIENCES,, WARSAW UNIVERSITY OF TECHNOLOGY, PL. POLITECHNIKI 1, 00-661 WARSAW, POLAND,
E-mail address: pawel.szablowski@gmail.com